## OPTION - A

Paper: MAT-HE-5016

# (Number Theory)

- 1. Answer the following questions as directed:  $1 \times 10 = 10$ 
  - (a) State Goldbach conjecture.
    - (b) If p and q are twin primes, then which of the following statements is true?

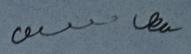
(i) 
$$pq = (p+1)^2 - 1$$

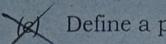
(ii) 
$$pq = (p+1)^2 + 1$$

(iii) 
$$pq = (p-1)^2 + 1$$

- (iv) None of the above
- Give an example to show that  $a^2 \equiv b^2 \pmod{n}$  need not imply that  $a \equiv b \pmod{n}$ .
- (d) State whether the following statement is True or False:

"The polynomial function  $f: N \to N$  defined by  $f(n) = n^2 + n + 41$  provides only prime numbers."





Define a pseudoprime number.

- (f) Find the sum of all positive divisors of 360.
- (g) Which of the following is a perfect number?
  - (i) 9
  - (ii) 10
  - (iii) 18
  - (jv) 28
- If x is not an integer then find the value of [x]+[-x].
- State whether the following statement is True or False:
  - "If  $\tau(n)$  is an odd integer, then  $\sqrt{(n)^{\tau(n)}}$  is not an integer."
- Find the number of integers less than 900 and prime to 900.
- 2. Answer the following questions:  $2 \times 5 = 10$ Find the remainder when  $41^{65}$  is divided by 7.

If  $a \equiv b \pmod{n}$ , then show that  $a - m \equiv b - m \pmod{n}$ , where m is any integer.

Show that any prime of the form 3k+1 is also of the form 6k+1, where k is an integer.

(d) If n is an odd positive integer, then prove that  $\phi(2n) = \phi(n)$ .

For 
$$n \ge 3$$
, evaluate  $\sum_{k=1}^{n} \mu(n!)$ 

3. Answer **any four** questions:  $5 \times 4 = 20$ 

- (a) Show that  $a \equiv b \pmod{n}$  if and only if a and b have same remainder on division by n.
- (b) Show that there are infinite number of primes of the form 4n + 3.

Solve using Chinese Remainder Theorem the simultaneous congruences:

 $x \equiv -2 \pmod{12}; x \equiv 6 \pmod{10}; x \equiv 1 \pmod{15}$ 

(d) If p is a prime and n is a positive integer, then show that the exponent e such that

$$p^e/n!$$
 is atmost  $\sum_{i=1}^{\infty} \left[ \frac{n}{p^i} \right]$ .

(e) Show that the system of linear congruences

$$ax + by \equiv r \pmod{n}$$
  
 $cx + dy \equiv s \pmod{n}$ 

has a unique solution modulo n whenever gcd(ad - bc, n) = 1.

- (f) If  $n \ge 1$  is an integer, then show that  $\sigma(n)$  is odd  $\Leftrightarrow n$  is a perfect square or twice a perfect square.
- 4. (a) State and prove Fermat's theorem. Is the converse of this theorem true? Justify your answer. 1+5+4=10

#### OR

(b) (i) Show that every integer n > 1, is either a perfect square or the product of a square-free integer and a perfect square.

 $n = a_m (1000)^m + a_{m-1} (1000)^{m-1} + ... + a_1 (1000) + a_0$ where  $a_k$ 's are integers such that

$$0 \le a_k \le 999$$
 and  $T = \sum_{k=0}^{m} (-1)^k a_k$ .

Prove that n is divisible by 7 if and only if T is divisible by 7.

- 5. (a) (i) If p is a prime then show that  $(p-1)! \equiv -1 \pmod{p}$ . Also verify it for p=13. 4+3=7
  - (ii) Show that any integer of the form  $8^n + 1$  is not a prime.

### OR

- (b) State and prove Fundamental Theorem of Arithmetic. Also find a prime number p such that 2p+1 and 4p+1 are also primes. 1+6+3=10
- 6. (a) (i) For each positive integer  $n \ge 1$ , prove that

$$\phi(n) = \sum_{d/n} \mu(d) \frac{n}{d} = n \prod_{p/n} \left(1 - \frac{1}{p}\right)$$

(ii) If n is the product of a pair of twin primes, prove that

$$\phi(n)\sigma(n) = (n+1)(n-3).$$

#### OR

(b) (i) If f is a multiplicative arithmetic function, then show that

$$g_1(n) = \sum_{d/n} f(d)$$
 and

$$g_2(n) = \sum_{d/n} \mu(d) f(d)$$

are both multiplicative arithmetic functions. 7

- (ii) If n is an even positive integer, then prove that  $\phi(2n) = 2\phi(n)$ .
- 7. (a) (i) Define Möbius pair. If (f,g) is a Möbius pair and either f or g is multiplicative then show that both f and g are multiplicative. 2+3=5
  - (ii) If p is a prime number and k is any positive integer, then show that

$$\phi(p^k) = p^k \left(1 - \frac{1}{p}\right).$$

- (b) (i) If x and y are real number then show that  $[x]+[y] \le [x+y] \le [x]+[y]+1.$  5
  - (ii) If  $n = p_1^{m_1} . p_2^{m_2} . ... . p_r^{m_r}$  where  $p_i$ 's are distinct primes and  $m_i \in \mathbb{N}, m_i \ge 1$  then for each  $r \ge 1$  prove that

$$\tau(n) = \prod_{i=1}^{r} (m_i + 1).$$

$$\frac{2^{4}-1}{2^{-1}} \cdot \frac{3^{3}-1}{3^{-1}} \cdot \frac{5^{2}-1}{5^{-1}}$$

$$\frac{2^{4}-1}{2^{-1}} \cdot \frac{3^{3}-1}{3^{-1}} \cdot \frac{5^{2}-1}{5^{-1}}$$

$$\frac{2^{5}-1}{2^{5}} \cdot \frac{2x^{1/3}}{x} \cdot \frac{2x^{1/3}}{x}$$