# OPTION – A

## Paper : MAT–HE–5016

## *(Number Theory)*

1. Answer the following questions as directed : $1 \times 10 = 10$

(a) State Goldbach conjecture.

(b) If $p$ and $q$ are twin primes, then which of the following statements is true ?

5, 3

(i) $pq = (p+1)^2 - 1$ $6^2 - 1 =$

(ii) $pq = (p+1)^2 + 1$

(iii) $pq = (p-1)^2 + 1$

(iv) None of the above

(c) Give an example to show that $a^2 \equiv b^2 \pmod{n}$ need not imply that $a \equiv b \pmod{n}$.

(d) State whether the following statement is True *or* False :

"The polynomial function $f : N \to N$ defined by $f(n) = n^2 + n + 41$ provides only prime numbers."

(e) Define a pseudoprime number.

(f) Find the sum of all positive divisors of 360.

(g) Which of the following is a perfect number?

(i) 9

(ii) 10

(iii) 18

(iv) 28

(h) If $x$ is not an integer then find the value of $[x]+[-x]$.

(i) State whether the following statement is True *or* False :

"If $\tau(n)$ is an odd integer, then $\sqrt{(n)^{\tau(n)}}$ is not an integer."

(j) Find the number of integers less than 900 and prime to 900.

2. Answer the following questions : $2\times5=10$

(a) Find the remainder when $41^{65}$ is divided by 7.

(b) If $a \equiv b \pmod{n}$, then show that $a - m \equiv b - m \pmod{n}$, where $m$ is any integer.

(c) Show that any prime of the form $3k+1$ is also of the form $6k+1$, where $k$ is an integer.

(d) If $n$ is an odd positive integer, then prove that $\phi(2n) = \phi(n)$.

(e) For $n \geq 3$, evaluate $\sum_{k=1}^{n} \mu(n!)$

3. Answer ***any four*** questions : 5×4=20

(a) Show that $a \equiv b \pmod{n}$ if and only if $a$ and $b$ have same remainder on division by $n$.

(b) Show that there are infinite number of primes of the form $4n+3$.

(c) Solve using Chinese Remainder Theorem the simultaneous congruences :

$$x \equiv -2 \pmod{12};\ x \equiv 6 \pmod{10};\ x \equiv 1 \pmod{15}$$

(d) If $p$ is a prime and $n$ is a positive integer, then show that the exponent $e$ such that

$$p^e / n! \text{ is atmost } \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right].$$

(e) Show that the system of linear congruences

$$ax + by \equiv r \pmod{n}$$
$$cx + dy \equiv s \pmod{n}$$

has a unique solution modulo $n$ whenever $\gcd(ad - bc, n) = 1$.

(f) If $n \geq 1$ is an integer, then show that $\sigma(n)$ is odd $\Leftrightarrow n$ is a perfect square or twice a perfect square.

4. (a) State and prove Fermat's theorem. Is the converse of this theorem true ? Justify your answer. 1+5+4=10

***OR***

(b) (i) Show that every integer $n > 1$, is either a perfect square or the product of a square-free integer and a perfect square. 5

(ii) Let

$$n = a_m (1000)^m + a_{m-1} (1000)^{m-1} + \dots + a_1 (1000) + a_0$$

where $a_k$'s are integers such that

$0 \le a_k \le 999$ and $T = \sum_{k=0}^{m} (-1)^k a_k$.

Prove that $n$ is divisible by 7 if and only if $T$ is divisible by 7. 5

5. (a) (i) If $p$ is a prime then show that $(p-1)! \equiv -1 \pmod{p}$. Also verify it for $p = 13$. 4+3=7

(ii) Show that any integer of the form $8^n + 1$ is not a prime. 3

**OR**

(b) State and prove Fundamental Theorem of Arithmetic. Also find a prime number $p$ such that $2p+1$ and $4p+1$ are also primes. 1+6+3=10

6. (a) (i) For each positive integer $n \ge 1$, prove that

$$\phi(n) = \sum_{d/n} \mu(d) \frac{n}{d} = n \prod_{p/n} \left(1 - \frac{1}{p}\right)$$

7

(ii) If $n$ is the product of a pair of twin primes, prove that

$$\phi(n)\sigma(n) = (n+1)(n-3). \qquad 3$$

**OR**

(b) (i) If $f$ is a multiplicative arithmetic function, then show that

$$g_1(n) = \sum_{d/n} f(d) \text{ and}$$

$$g_2(n) = \sum_{d/n} \mu(d) f(d)$$

are both multiplicative arithmetic functions. 7

(ii) If $n$ is an even positive integer, then prove that $\phi(2n) = 2\phi(n)$. 3

7. (a) (i) Define Möbius pair. If $(f, g)$ is a Möbius pair and either $f$ or $g$ is multiplicative then show that both $f$ and $g$ are multiplicative. 2+3=5

(ii) If $p$ is a prime number and $k$ is any positive integer, then show that

$$\phi\left(p^k\right) = p^k\left(1 - \frac{1}{p}\right). \qquad 5$$

**OR**

(b) (i) If $x$ and $y$ are real number then show that

$$[x]+[y] \le [x+y] \le [x]+[y]+1. \qquad 5$$

(ii) If $n = p_1^{m_1} . p_2^{m_2} . \dots . p_r^{m_r}$ where $p_i$'s are distinct primes and $m_i \in N,\ m_i \ge 1$ then for each $r \ge 1$ prove that

$$\tau(n) = \prod_{i=1}^{r} (m_i + 1). \qquad 5$$

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15
13
195
6
1170

$$\frac{2^4-1}{2-1} \cdot \frac{3^3-1}{3-1} \cdot \frac{5^2-1}{5-1}$$

$$= \frac{15}{1} \cdot \frac{26}{2}^{13} \cdot \frac{24}{4}^{6}$$