(For Old Syllabus)

(Riemann Integration and Metric Spaces)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer the following questions : $1\times10=10$

   (a) Describe an open ball on the real line $\mathbb{R}$ for the usual metric d.

   (b) Find the limit point of the set

   $$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}.$$

   (c) Define Cauchy sequence in a metric space (X, d).

   (d) Let A and B be two subsets of a metric space (X, d). Then

   (i) $(A \cap B)^0 = A^0 \cap B^0$

   (ii) $(A \cup B)^0 = A^0 \cup B^0$

   (iii) $(A \cap B)' = A' \cap B'$

   (iv) $(A \cup B)' = A' \cup B'$

   where A^0 denotes interior of A

   A' denotes derived set of A

   (Choose the correct answer)

(e) In a complete metric space

(i) every sequence is bounded

(ii) every bounded sequence is convergent

(iii) every convergent sequence is bounded

(iv) every Cauchy sequence is convergent

(Choose the correct answer)

(f) Let $\{F_n\}$ be a decreasing sequence of closed subsets of a complete metric space and $d(F_n) \to 0$ as $n \to \infty$. Then

(i) $\bigcap_{n=1}^{\infty} F_n = \phi$

(ii) $\bigcap_{n=1}^{\infty} F_n$ contains at least one point

(iii) $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point

(iv) $d\left(\bigcap_{n=1}^{\infty} F_n\right) > 0$

(Choose the correct answer)

(g) Let (X, d) and (Y, ρ) be metric spaces and $A \subset X$. Let $f : X \to Y$ be continuous on X. Then

(i) $f(A) = f(\overline{A})$

(ii) $f(\overline{A}) \subset \overline{f(A)}$

(iii) $\overline{f(A)} \subset f(\overline{A})$

(iv) $f(A) = f(A^0)$

(Choose the correct answer)

(h) What is meant by partition P of an interval $[a, b]$?

(i) Prove that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

(j) Define the upper and the lower Darboux sums of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition P.

2. Answer the following questions : $2 \times 5 = 10$

(a) Prove that in a discrete metric space every singleton set is open.

(b) For any two subsets F_1 and F_2 of a metric space (X, d), prove that

$$\overline{(F_1 \cup F_2)}. = \overline{F_1} \cup \overline{F_2}$$

(c) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Then if f is continuous on X, prove that

$\overline{f^{-1}(B)} \subset f^{-1}\left(\overline{B}\right)$ for all subsets B of Y.

(d) Find $L(f, P)$ and $U(f, P)$ for a constant function $f : [a, b] \to \mathbb{R}$.

(e) Examine the existence of improper integral $\int_0^1 \frac{1}{\sqrt{x}} dx$.

3. Answer ***any four*** parts : 5×4=20

(a) Let d be a metric on the non-empty set X. Prove that the function d' defined by

$d'(x, y) = min\{1, d(x, y)\}$

where $x, y \in X$ is a metric on X. State whether d' is bounded or not.

4+1=5

Contd.

(b) In a metric space (X, d), prove that every closed sphere is a closed set.

(c) Prove that if a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, then the sequence also converges to the same limit as the subsequence.

(d) Let (X, d) be a metric space and let $\{Y_\lambda, \lambda \in l\}$ be a family of connected sets in (X, d) having a non-empty intersection. Then prove that $Y = \bigcup_{\lambda \in l} Y_\lambda$ is connected.

(e) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{otherwise} \end{cases}$

Prove that f is not integrable on $[0, 1]$.

(f) Let $f : [a, b] \to R$ be bounded and monotone. Prove that f is integrable.

4. Answer ***any four*** parts : 10×4=40

(a) (i) Define a metric space.

Let

$$X = \mathbb{R}^n = \{x = (x_1, x_2, \ldots x_n), x_i \in \mathbb{R}, 1 \le i \le n\}$$

be the set of all real n-tuples.

For $x = (x_1, x_2, \ldots, x_n)$ and

$y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ define

$$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}.$$

Prove that $(\mathbb{R}^n, d)$ is a metric space. 2+4=6

(ii) Prove that in a metric space (X, d), a finite intersection of open sets is open. 4

(b) Let Y be a subspace of a metric space (X, d). Prove the following : 5+5=10

(i) Every subset of Y that is open in Y is also open in X if and only if Y is open in X.

(ii) Every subset of Y that is closed in Y is also closed in X if and only if Y is closed in X.

(c) (i) Prove that the function $f:[0,1]\to\mathbb{R}$ defined by $f(x)=x^2$ is uniformly continuous. Further prove that the function will not be uniformly continuous if the domain is $\mathbb{R}$. 3+3=6

(ii) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces and let $f: X\to Y$ and $g: Y\to Z$ be continuous. Prove that the composition $g\circ f$ is a continuous map of X into Z. 4

(d) When a metric space is said to be disconnected?

Prove that a metric space (X, d) is disconnected if and only if there exists a non-empty proper subset of X which is both open and closed in (X, d). 2+8=10

(e) (i) Show that the metric space (X, d) where X denotes the space of all sequences $x = (x_1, x_2, x_3, \ldots, x_n)$ of real numbers for which

$$\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} < \infty \ (p \geq 1)$$ and d is the

metric given by

$$d_p(x, y) = \left(\sum_{k=1}^{\infty} (x_k - y_k)^p\right)^{1/p}, \quad x, y \in X$$

is a complete metric space. 7

(ii) Let X be any non-empty set and let d be defined by

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y \\ 1 & \text{if} \quad x \neq y \end{cases}$$

Show that (X, d) is a complete metric space. 3

(f) Prove that a bounded function $f:[a, b] \to R$ is integrable if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

(g) Let $f:[0, 1] \to \mathbb{R}$ be continuous. Let $C_i \in \left[\frac{i-1}{n}, \frac{i}{n}\right], n \in N$. Then prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(C_i) = \int_0^1 f(x)\, dx.$$

Using it, prove that $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \frac{\pi}{4}$.

6+4=10

(h) (i) Prove that a mapping $f: X \to Y$ is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y. 5

(ii) Let f and g be continuous on $[a, b]$. Also assume that g does not change sign on $[a, b]$. Then prove that for some $c \in [a, b]$ we have

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

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